

1. Formulation of the Problem. It was shown in [1] that there is more than one correct mathematical formulation for a Lekhnitskii problem concerning the deformation of an elastic half-space with a cylindrical shaft under its own weight. (We should also point out [4], where the analogous problem was examined for the case of an inclined shaft.) The selection of the conditions at infinity should correspond to the physical reality of the problem. Since an infinite half-space is an idealization of a large but finite elastic body, specific mechanical problems should be subjected to asymptotic analysis. Here, we study the deformation of finite bodies with long thin shafts under the influence of body forces. These bodies are as follows: a gravitating sphere (a) (Fig. 1); and a weighable thick slab whose base (b) lays on a smooth rigid surface (c) bound to a perfectly rigid half-space or (d) does not have supports (Fig. 2).

The studies [2, 3] noted the existence of the following solution to this problem in an infinite region (Fig. 3):

$$u_r^{(1)}(\mathbf{x}) = \nu h^2 \gamma z [2(1-\nu)\mu r]^{-1}, \quad u_z^{(1)}(\mathbf{x}) = (1-2\nu)\gamma z^2 [4(1-\nu)\mu]^{-1} + \nu \gamma h^2 [2(1-\nu)\mu]^{-1} \ln(rh^{-1}); \quad (1.1)$$

$$\sigma_{rr}(\mathbf{u}^{(1)}; \mathbf{x}) = \frac{\nu \gamma z}{1-\nu} \left(1 - \frac{h^2}{r^2}\right), \quad \sigma_{\varphi\varphi}(\mathbf{u}^{(1)}; \mathbf{x}) = \frac{\nu \gamma z}{1-\nu} \left(1 + \frac{h^2}{r^2}\right), \quad (1.2)$$

$$\sigma_{zz}(\mathbf{u}^{(1)}; \mathbf{x}) = \gamma z, \quad \sigma_{rz}(\mathbf{u}^{(1)}; \mathbf{x}) = 0.$$

Another solution, found in [1]:

$$u_r^{(2)} = -\nu [2(1+\nu)\mu]^{-1} \gamma z r, \quad u_z^{(2)} = \gamma [4(1+\nu)\mu]^{-1} (z^2 + \nu r^2); \quad (1.3)$$

$$\sigma_{zz}(\mathbf{u}^{(2)}; \mathbf{x}) = \gamma z, \quad \sigma_{rr}(\mathbf{u}^{(2)}; \mathbf{x}) = \sigma_{\varphi\varphi}(\mathbf{u}^{(2)}; \mathbf{x}) = \sigma_{rz}(\mathbf{u}^{(2)}; \mathbf{x}) = 0. \quad (1.4)$$

Both solutions have obvious flaws. The component $u_r^{(2)}$ of field (1.3) undergoes significant growth at $r \rightarrow \infty$. Solution (1.1) does not have this shortcoming, but in (1.2) the stresses σ_{rr} and $\sigma_{\varphi\varphi}$ do not decay at $r \rightarrow \infty$, while at $|z| > 2\gamma^{-1}\mu(\nu^{-1} + 1)$, we observe interpenetration of the walls of the shaft (i.e., $u_r < -h$ at $r = h$). It should be emphasized that in Eqs. (1.3) and (1.4), with an increase in depth, deformation leads to an increase in the cross section of the shaft. These disparities mean that the region of application of the problem of an infinite body is limited. Thus, the solutions of all of the problems will henceforth be compared only near the outlet hole of the shaft.

The asymptote of the solution of the problem of a shaft in a finite body is determined by the Lekhnitskii solution only in situation "c." In the other three problems, the principal term of the asymptotic solution consists of the following displacement and stress fields:

$$u_r = \sigma_{\infty} (2\mu)^{-1} [(1-\nu)(1+\nu)^{-1} r + h^2 r^{-1}], \quad (1.5)$$

$$u_z = -\sigma_{\infty} \nu z \mu^{-1} (1+\nu)^{-1}, \quad u_{\varphi} = 0; \quad (1.5)$$

$$\sigma_{rr} = \sigma_{\infty} (1 - h^2 r^{-2}), \quad \sigma_{\varphi\varphi} = \sigma_{\infty} (1 + h^2 r^{-2}) \quad (1.6)$$

[see Eqs. (2.4), (3.8), and (5.7)]. These fields correspond to the axisymmetric tension (compression, at $\sigma_{\infty} < 0$) of a half-space with a cylindrical shaft. Thus, for unlimited expansion of the region, there is no correct approach to the Lekhnitskii problem in cases "a", "b", and "d."

The very presence of solution (1.5)-(1.6) in the representation of the stress-strain state near the outlet hole of the shaft is not unexpected: it should be pointed out that the relative contribution of the fields (1.5)-(1.6) to the asymptotic solution increases

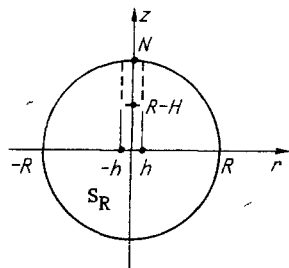


Fig. 1

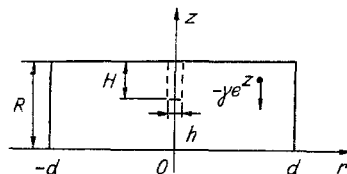


Fig. 2

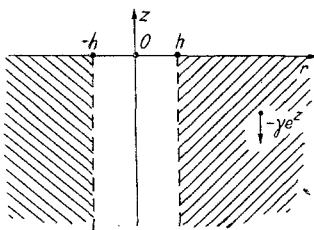


Fig. 3

with an increase in the radius R of the sphere or the radius d and thickness R of the slab. It is a well-known fact that the stresses in horizontal planes predominate in rock mechanics [5, 6]; wells and shafts are designed not on the basis of the Lekhnitskii formulas, but with allowance for "coefficients of lateral stresses and thrust."

In Part 3 we check the following fact: if a compressive load of intensity p is applied to the lateral surface of a slab with a shaft on a smooth surface, then the principal term of (1.5)-(1.6) asymptotically vanishes at a certain value of p . The minor term of the asymptotic solution coincides with the Lekhnitskii solution (1.1)-(1.2). However, no such coincidence occurs in problems "a" and "d." In the case of the gravitating sphere, this fact is connected with the curvature of the surface (which in turn results from the fact that the correction terms for solution (1.5)-(1.6) are of the same order of magnitude as the asymptotes of the field (1.1)-(1.2) present in the minor term. The minor asymptotic term for a slab with an unsupported base (problem "d") is a linear combination of the solutions (1.1)-(1.2) and (1.3)-(1.4) of the Lekhnitskii problem.

Many investigations have dealt with the formulations of problems of the theory of elasticity for infinite bodies (see [7, 8], etc.). Special difficulties arise in those problems which involve infinite energy intervals. Basic postulates on the existence and uniqueness of solutions cease to be valid in such cases, and a careless treatment entailing additional conditions at infinity leads to erroneous conclusions and various types of paradoxes [8, examples in Part 7 of Chapter 3]. The Lekhnitskii problem being examined here belongs to this class of problems.

The asymptotic analysis performed in the present study permits the conclusion that the Lekhnitskii formulas are valid only for certain interpretations among the large number of possible interpretations of the problem in the infinite region. Specifically, the formulas are to be regarded as the limit of problems for large but finite elastic bodies.

Direct passage to the limit in the infinite region in other problems is either incorrect or leads to solutions different from the Lekhnitskii solution.

2. Gravitating Sphere with a Shaft. Let S_R be a sphere of radius R with center O . The sphere is filled with an elastic material having the Lamé constants λ and μ . The quantities g and γ are acceleration due to gravity and the specific weight of the material on the surface. We will use C_h to denote a cylinder $\{x : r \leq h\}$ with a small radius h ; here, (r, φ, z) are cylindrical coordinates. We set $\Omega = S_R \setminus \bar{G}$, $G = \{x \in \bar{C}_h : z > R - H\}$ (H is the depth of the shaft). We will assume that $R \gg H \gg h$. We will examine the problem of the deformation of the body Ω under the influence of the gravitational field

$$L(\partial/\partial x)u(x) + F(x) = 0, \quad x \in \Omega; \quad \sigma^{(n)}(u; x) = 0, \quad x \in \partial\Omega, \quad (2.1)$$

where \mathbf{L} is an operator in the Lamb system; \mathbf{u} is the displacement vector; $\boldsymbol{\sigma}(\mathbf{u})$ is the stress tensor; \mathbf{n} is the vector of a unit outward normal; $\boldsymbol{\sigma}^{(n)} = \boldsymbol{\sigma} \cdot \mathbf{n}$; $\mathbf{F}(\mathbf{x}) = -g\rho R^{-1} \mathbf{e}^\rho$; \mathbf{e}^ρ is the unit vector; (ρ, θ, φ) are spherical coordinates. Problem (2.1) should be treated as a problem in a region with a singularly disturbed boundary. According to [9], far from the shaft G the field \mathbf{u} is described mainly by the solution \mathbf{v} of the problem of a solid gravitating sphere. This solution has the form

$$u_\rho = [10(\lambda + 2\mu)R]^{-1} \gamma \rho [\rho^2 - R^2(6\mu + 5\lambda)(2\mu + 3\lambda)^{-1}]; \quad (2.2)$$

$$\sigma_{\rho\rho} = -\gamma [10(\lambda + 2\mu)R]^{-1} (6\mu + 5\lambda)(R^2 - \rho^2), \quad (2.3)$$

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = -\gamma [10(\lambda + 2\mu)R]^{-1} [(6\mu + 5\lambda)(R^2 - \rho^2) + 4\mu\rho^2].$$

Here and below, the components of the fields that are equal to zero are not indicated. The vector \mathbf{v} leaves a closure error in the boundary conditions for $\partial G \cap \partial\Omega$ from (2.1). Thus [9], boundary layers exist near ∂G : a two-dimensional boundary layer for the middle part of the shaft and two three-dimensional boundary layers for its end and outlet hole. Since we want to compare the solution of problem (2.1) with the Lekhnitskii solution, we will restrict ourselves to study of the last boundary layer.

We introduce "stretched" dimensionless coordinates $\xi = h^{-1}(\mathbf{x} - \mathbf{N})$, where \mathbf{x} represents cartesian coordinates and $\mathbf{N} = (0, 0, R)$ is the north pole of the sphere ∂S_R . Considering the relationship between the dimensions R , H , and h , we conclude that in the coordinates ξ (after the formal transition to $h = 0$) the region Ω is transformed into a half-space $R_-^3 = \{\xi : \xi_3 < 0\}$ with an infinite shaft C_1 . In accordance with (2.3), the error of the field \mathbf{v} in the boundary condition on the lateral surface G is given by the formula

$$\sigma_{rr}(\mathbf{v}; \mathbf{x}) = -[5(\lambda + 2\mu)]^{-1} 2\mu\gamma R(1 + O(hR^{-1}|\xi|)). \quad (2.4)$$

Thus, the leading term of the boundary layer $h\mathbf{w}^0(\xi)$ is the solution of the elastic problem in $R_-^3 \setminus C_1$ under the condition that on the lateral surface we assign a constant normal stress

$$\mathbf{w}^0(\xi) = -[5(\lambda + 2\mu)]^{-1} \gamma R \mathbf{X}^{(0)}; \quad (2.5)$$

$$X_r^{(0)}(\xi) = (\xi_1^2 + \xi_2^2)^{-1/2}, \quad X_\varphi^{(0)} = X_z^{(0)} = 0. \quad (2.6)$$

3. Circular Plate with a Shaft on a Smooth Rigid Base. Let $\Pi = \{\mathbf{x} \in c_d : 0 < z < R\}$, $\Omega = \Pi \setminus \overline{G}$ (G is the set from Part 2, d is the radius of the plate, and R is its thickness, $d \gg R \gg H \gg h$). We will examine the problem of the deformation of a body Ω under its own weight γe^3 and compressive radial forces p applied to the lateral surface S of the slab Π . The slab lays on a perfectly rigid smooth base. The upper part of the slab is free of stresses. The displacement vector \mathbf{u} is satisfied by the equations

$$\mathbf{L}(\partial/\partial \mathbf{x})\mathbf{u}(\mathbf{x}) - \gamma e^{(3)} = \mathbf{0}, \quad \mathbf{x} \in \Omega; \quad (3.1)$$

$$\boldsymbol{\sigma}^{(3)}(\mathbf{u}; \mathbf{y}, R) = \mathbf{0}, \quad h < |\mathbf{y}| < d; \quad (3.2)$$

$$u_3(\mathbf{y}, 0) = 0, \quad \sigma_{j3}(\mathbf{u}; \mathbf{y}, 0) = 0, \quad j = 1, 2, \quad |\mathbf{y}| < d; \quad (3.3)$$

$$\boldsymbol{\sigma}^{(r)}(\mathbf{u}; \mathbf{y}, z) = -p e^{(r)}, \quad |\mathbf{y}| = d, \quad z \in (0, R); \quad (3.4)$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}; \mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \partial G \cap \Omega. \quad (3.5)$$

Here, $\mathbf{y} = (x_1, x_2)$. The particular solution \mathbf{v}^0 of the problem of a weighable layer [$v_r^0 = v_\varphi^0 = 0$, $v_z^0(\mathbf{x}) = (1/2)\gamma(2\mu + \lambda)^{-1}z(z - 2R)$] eliminates the inhomogeneity in equilibrium equations (3.1) and satisfies (3.2)-(3.3) but leaves an error in boundary conditions (3.4)-(3.5). Using the smallness of the ratio R/d , we account for conditions on the lateral surface of the slab by means of the solution $\Psi = (\Psi_r, 0)$ of the problem of an axisymmetric generalized plane-stress state in a circle $\{\mathbf{y} : |\mathbf{y}| < d\}$ with the boundary condition

$$2\mu \frac{\partial \Psi_r}{\partial r} + \frac{2\mu\lambda}{2\mu + \lambda} \left(\frac{\partial \Psi_r}{\partial r} + r^{-1}\Psi_r \right) = \frac{1}{2} \frac{\lambda}{2\mu + \lambda} R\gamma - p \quad \text{at } r = d. \quad (3.6)$$

It is clear that $\Psi_r = [4\mu(2\mu + 3\lambda)]^{-1}(\lambda R\gamma - 2(2\mu + \lambda)p)r$. The components of the three-dimensional displacement vector \mathbf{v}^1 beyond the circumference of the lateral surface S of the slab are established by means of the equalities

$$v_r^1 = \Psi_r, \quad v_\varphi^1 = 0, \quad v_z^1 = -vz(1 - \nu)^{-1}(\partial \Psi_r / \partial r + r^{-1}\Psi_r). \quad (3.7)$$

Let us explain Eqs. (3.6) and (3.7). Equations (3.7) are the normal equations for a plane stress state in a slab of thickness $2R$. We change over from Eqs. (3.1)-(3.3) to the problem of the deformation of the slab $\{x \in c_d : |z| < R\}$ by an even-numbered continuation of u_r and u_φ and an odd-numbered continuation of u_z in R_-^3 . Equality (3.6) means that the error of the sum $v^0 + v^1$ in boundary condition (3.4) on the generatrix of the cylindrical surface S has a zero principle vector. It is known that a two-dimensional boundary layer forms at the edge of the slab and that this layer decays exponentially into the slab due to the above-indicated self-balancing. Thus, near G , $v^0 + v^1$ approximates the solution of the problem of a solid slab Π to within $O(Rd^{-1})$.

As in Part 2, boundary layers are formed near G ; they compensate for the error of the field $v^0 + v^1$ in conditions (3.5). The dominant term of the boundary layer, associated with the outlet hole of the shaft, has the form $hw^0(\xi)$, where $\xi = h^{-1}(x - N)$ are the "stretched" variables,

$$w^{(0)}(\xi) = (\lambda R \gamma - 2(2\mu + \lambda)p) [4\mu(2\mu + \lambda)]^{-1} X^{(0)}(\xi), \quad (3.8)$$

$X^{(0)}$ is the vector (2.6). This solution compensates for the error $(1/2)(2\mu + \lambda)^{-1}(\lambda R \gamma - 2(2\mu + \lambda)p)$ in the boundary condition $\sigma_{rr} = 0$ on the lateral surface of the shaft. If

$$\lambda R \gamma - 2(2\mu + \lambda)p = 0, \quad (3.9)$$

then $w^0 = 0$, and we must calculate the second term of the boundary layer $h^2 w^1(\xi)$. This term is determined by the formula

$$w^1(\xi) = v(1 - v)^{-1} X^{(1)}(\xi); \quad (3.10)$$

$$X_r^{(1)} = \xi_3 \gamma [2\mu(\xi_1^2 + \xi_2^2)^{1/2}]^{-1}, \quad X_z^{(1)} = -\gamma(4\mu)^{-1} \ln(\xi_1^2 + \xi_2^2), \quad X_\varphi^{(1)} = 0. \quad (3.11)$$

The solution (3.10) compensates for the error left by the field v^0 in the boundary condition $\sigma_{rr} = 0$ for ∂G . It is not hard to see that to within the rigid displacement in the direction of the Oz axis, the sum $v^0 + h^2 w^1$ near the outlet hole coincides with the Lekhnitskii solution. If condition (3.9) is not satisfied, then there is no such coincidence.

4. Circular Slab with a Shaft Rigidly Fixed on the Lower Base. We will keep the notation from Part 3 and examine the problem of the deformation of a slab with a shaft in the case of complete bonding of the lower base to a perfectly rigid half-space R_-^3 . The mathematical formulation of the problem reduces to system (3.1) with boundary conditions (3.2), (3.4), (3.5), and

$$u(y, 0) = 0, \quad |y| < d. \quad (4.1)$$

The scheme used to study this problem differs little from that described in Part 3. The simplifications made are connected with the absence of the term v^1 in the smooth solution. In fact, the field v^0 leaves an error in boundary condition (3.4), which is compensated for by the two-dimensional boundary layer around S . This boundary layer is the solution of the plane problem on the deformation of a half-strip having one side which is rigidly fixed, by virtue of (4.1). Thus, it decays exponentially, while the field (3.7) and the boundary layer $hw^0(\xi)$ are absent. We ultimately find that the three-dimensional boundary layer has the form $h^2 w^1(\xi)$, where w^1 is the vector (3.10). As in Part 3, the sum $v^0 + h^2 w^1$ gives the Lekhnitskii solution.

5. Shaft in a Circular Slab Not Having Supports. We will examine the same problem as in Parts 3 and 4, but we will assume that the lower base is free of stresses (the slab "hangs" above a cavity), while the lateral surface is rigidly fixed

$$\sigma^{(3)}(u; y, 0) = 0, \quad |y| < d; \quad (5.1)$$

$$u(y, z) = 0, \quad |y| = d, \quad z \in (0, R) \quad (5.2)$$

[compare with (3.3), (4.1), and (3.4)]. As the dominant term of the asymptote of the solution of problem (3.1)-(3.2), (5.1)-(5.2), (3.5), we take the solution v^0 of the problem of the bending of a plate

$$v_j^0(x) = -R\zeta \frac{\partial \psi}{\partial y_j}(y) + R^3 \left[-\frac{\zeta^3}{6} \frac{v-2}{1-v} + \zeta \frac{v-6}{24(1-v)} \right] \Delta_y \frac{\partial \psi}{\partial y_j}(y),$$

$$j = 1, 2;$$

$$v_3^0(x) = \psi(y) + R^2 \frac{v}{1-v} \left(\frac{\zeta^2}{2} - \frac{1}{24} \right) \Delta_y \psi(y). \quad (5.4)$$

Here, $\zeta = R^{-1}z - 1/2$ is the coordinate of the internal boundary layer in a thin cylindrical region (the ratio R/d is a small quantity). The function

$$\psi(y) = -(32\mu)^{-1} 3\gamma(1-v)(dR^{-1})^4(1-(rd^{-1})^2)^2 \quad (5.5)$$

is the solution of the problem in the Kirchhoff theory of plate bending

$$\Delta_y^2 \psi(r) = -6\gamma(1-v)\mu^{-1}R^{-4}, \quad |r| < d, \quad \psi(d) = 0, \quad \partial\psi/\partial r(d) = 0. \quad (5.6)$$

It should be noted that, as in Parts 3 and 4, the boundary conditions in problem (5.6) are obtained by examining the problem of a two-dimensional boundary layer.

The field (5.3)-(5.4) leaves an error in Eq. (3.5). In accordance with (5.5), to within the minor terms, this error is given by the relations $\sigma_{rr} = (3/8)\gamma(1+v)d^2R^{-4}(R-2\xi_3h)$, $h\xi_3 = (1/2)R - \zeta R$, $h\xi_1 = y_2$, $h\xi_2 = y_1$ and is compensated for by means of the boundary layer $hw^{(0)}(\xi) + h^2w^{(1)}(\xi)$, where

$$w^{(0)}(\xi) = (3/16)\gamma\mu^{-1}(1+v)d^2R^{-3}X^{(0)}(\xi); \quad (5.7)$$

$$w^{(1)}(\xi) = (3/4)(1+v)d^2R^{-3}X^{(1)}(\xi). \quad (5.8)$$

As in the problem in Part 3, when condition (3.9) is violated near the outlet hole of the shaft, the solution of problem (3.1)-(3.2), (5.1)-(5.2) is determined mainly in a field corresponding to the compression of a half-space with a shaft [see (5.7) and compare with (3.8)]. The second term of the boundary layer, in contrast to Part 3, does not conform to the Lekhnitskii solution. It is connected with the following linear combination of solutions (1.1)-(1.2) and (1.3)-(1.4) discussed in [1]:

$$(3/4)d^2(1-v^2)(vR^4)^{-1}\{u^{(2)}(\xi) - u^{(1)}(\xi)\}. \quad (5.9)$$

It should be noted that, in accordance with (1.1), (1.3), expression (5.9) does not have a singularity at $v \rightarrow 0$. The field (5.9) coincides to within the multiplier with the solution U presented in [1] for a homogeneous problem in a half-space with a vertical shaft.

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